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Some kinds of $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy filters of *BL*-algebras

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ABSTRACT

The concepts of $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy (implicative, positive implicative and fantastic) filters of *BL*-algebras are introduced and some related properties are investigated. Some characterizations of these generalized fuzzy filters are derived. In particular, we describe the relationships among ordinary fuzzy (implicative, positive implicative and fantastic) filters, $(\in, \in \vee \mathrm{q})$-fuzzy (implicative, positive implicative and fantastic) filters and $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy (implicative, positive implicative and fantastic) filters of *BL*-algebras. Finally, we prove that a fuzzy set F of a *BL*-algebra L is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy implicative filter of L if and only if it is both an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy positive implicative filter and an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy fantastic filter.

1. Introduction and preliminaries

It is well known that certain information processing, especially inferences based on certain information, is based on classical two-valued logic. In making inference levels, it is natural and necessary to attempt to establish some rational logic system as the logical foundation for uncertain information processing. It is evident that this kind of logic cannot be two-valued logic itself but might form a certain extension of two-valued logic. Various kinds of non-classical logic systems have therefore been extensively researched in order to construct natural and efficient inference systems to deal with uncertainty.

Logic appears in a 'sacred' form (resp., a 'profane') which is dominant in proof theory (resp., model theory). The role of logic in mathematics and computer science is twofold — as a tool for applications in both areas, and a technique for laying the foundations. Non-classical logic – including many-valued logic, fuzzy logic, etc. – takes the advantage of the classical logic to handle information with various facets of uncertainty [1], such as fuzziness, randomness, and so on. Non-classical logic has become a formal and useful tool for computer science to deal with fuzzy information and uncertain information. Fuzziness and incomparability are two kinds of uncertainties often associated with human's intelligent activities in the real world, and they exist not only in the processed object itself, but also in the course of the object being dealt with.

The concept of *BL*-algebras was introduced by Hájek's as the algebraic structures for his Basic Logic [2]. A well known example of a *BL*-algebra is the interval [0, 1] endowed with the structure induced by a continuous t-norm. On the other hand, the *MV*-algebras, introduced by Chang in 1958 (see [3]), are one of the most well known classes of *BL*-algebras. In order to investigate the logic system whose semantic truth-value is given by a lattice, Xu [4] proposed the concept of lattice implication algebras and studied the properties of filters in such algebras [5]. Later on, Wang [6] proved that the lattice implication algebras are categorically equivalent to the *MV*-algebras. Furthermore, in order to provide an algebraic proof of

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the completeness theorem of a formal deductive system [7], Wang [8] introduced the concept of R_0-algebras. In fact, the MV-algebras, Gödel algebras and product algebras are the most known classes of BL-algebras. BL-algebras are further discussed by many researchers, see [9–21]. Recent investigations are concerned with non-commutative generalizations for these structures. In [22], Georgescu et al. introduced the concept of pseudo MV-algebras as a non-commutative generalization of MV-algebras. Several researchers discussed the properties of pseudo MV-algebras, see [23–27]. Pseudo BL-algebras are a common extension of BL-algebras and pseudo MV-algebras, see [28–33]. These structures seem to be a very general algebraic concept with the aim of expressing the non-commutative reasoning.

After the introduction of fuzzy sets by Zadeh [34], there have been a number of generalizations of this fundamental concept. A new type of fuzzy subgroup, that is, the $(\in, \in\vee \mathrm{q})$-fuzzy subgroup, was introduced in an earlier paper of Bhakat and Das [35,36] by using the combined notions of "belongingness" and "quasicoincidence" of fuzzy points and fuzzy sets, which was introduced by Pu and Liu [37]. In fact, the $(\in, \in\vee \mathrm{q})$-fuzzy subgroup is an important generalization of Rosenfeld's fuzzy subgroup. It is now natural to investigate similar type of generalizations of the existing fuzzy subsystems with other algebraic structures. With this objective in mind, Davvaz [38] applied this theory to near-rings and obtained some useful results. Further, Davvaz and Corsini [39] redefined fuzzy H_v-submodule and many valued implications. In [40], Zhan et al. also discussed the properties of interval-valued $(\in, \in\vee \mathrm{q})$-fuzzy hyperideals in hypernear-rings. For more details, the reader is referred to [35–41].

In [13], we introduced the concepts of $(\in, \in \vee q)$-fuzzy (implicative, positive implicative and fantastic) filters in BL-algebras and investigated some related properties. As a continuation of that paper, we further discuss the topic in this paper. In Section 2, we describe the relationships among ordinary fuzzy filters, $(\in, \in\vee \mathrm{q})$-fuzzy filters and $(\overline{\in}, \overline{\in}\vee \overline{q})$-fuzzy filters of BL-algebras. In Section 3, we divide things into three subsections. In Section 3.1, we describe the relationships among ordinary fuzzy implicative filters, $(\in, \in\vee \mathrm{q})$-fuzzy implicative filters and $(\overline{\in}, \overline{\in}\vee \overline{q})$-fuzzy implicative filters of BL-algebras. In Section 3.2, we describe the relationships among ordinary fuzzy positive implicative filters, $(\in, \in\vee \mathrm{q})$-fuzzy positive implicative filters and $(\overline{\in}, \overline{\in}\vee \overline{q})$-fuzzy positive implicative filters of BL-algebras. Further, the relationships among ordinary fuzzy fantastic filters, $(\in, \in\vee \mathrm{q})$-fuzzy fantastic filters and $(\overline{\in}, \overline{\in}\vee \overline{q})$-fuzzy fantastic filters of BL-algebras are considered in Section 3.3. Finally, in Section 4, we prove that a fuzzy set F of a BL-algebra L is an $(\overline{\in}, \overline{\in}\vee \overline{q})$-fuzzy implicative filter of L if and only if it is both $(\overline{\in}, \overline{\in}\vee \overline{q})$-fuzzy positive implicative filter and an $(\overline{\in}, \overline{\in}\vee \overline{q})$-fuzzy fantastic filter.

Recall that an algebra $L = (L, \leq, \wedge, \vee, \odot, \to, 0, 1)$ is a BL-algebra if it is a bounded lattice such that the following conditions are satisfied:

(i) $(L, \odot, 1)$ is a commutative monoid,

(ii) $\odot$ and $\to$ form an adjoin pair, i.e., $z \leq x \to y$ if and only if $x \odot z \leq y$ for all $x, y, z \in L$,

(iii) $x \wedge y = x \odot (x \to y)$,

(iv) $(x \to y) \vee (y \to x) = 1$.

In what follows, L is a BL-algebra unless otherwise specified.

In any BL-algebra L, the following statements are true (see [16]):

(1) $x \leq y \Leftrightarrow x \to y = 1$,
(2) $x \to (y \to z) = (x \odot y) \to z = y \to (x \to z)$,
(3) $x \odot y \leq x \wedge y$,
(4) $x \to y \leq (z \to x) \to (z \to y), x \to y \leq (y \to z) \to (x \to z)$.
(5) $x \to x' = x'' \to x$,
(6) $x \vee x' = 1 \Rightarrow x \wedge x' = 0$,
(7) $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$,

where $x' = x \to 0$.

A non-empty subset A of L is called a *filter* of L if it satisfies the following conditions: (i) $1 \in A$; (ii) $\forall x \in A, y \in L, x \to y \in A \Rightarrow y \in A$. It is easy to check that a non-empty subset A of L is a filter of L if and only if it satisfies: (i) $\forall x, y \in L, x \odot y \in A$; (ii) $\forall x \in A, x \leq y \Rightarrow y \in A$(see [15–18]).

Now, we call a filter A of L an *implicative filter* of L if it satisfies $x \to (z' \to y) \in A, y \to z \in A \Rightarrow x \to z \in A$. A filter A of L is said to be a *positive implicative filter* of L if it satisfies $x \to (y \to z) \in A, x \to y \in A \Rightarrow x \to z \in A$. A filter A of L is called a *fantastic filter* of L if it satisfies $z \to (y \to x) \in A, z \in A \Rightarrow ((x \to y) \to y) \to x \in A$ (see [11–13]).

We now review some fuzzy logic concepts. A fuzzy set of L is a function $F : L \to [0, 1]$ (see [34]).

Definition 1.1 (*[11]*). A fuzzy set F of L is called a *fuzzy filter* of L if it satisfies:

(F1) $\forall x, y \in L, F(x \odot y) \geq \min\{F(x), F(y)\}$,

(F2) F is order-preserving, that is, $\forall x, y \in L, x \leq y \Rightarrow F(x) \leq F(y)$.

Definition 1.2 (*[12,13]*). (i) A fuzzy filter F of L is called a *fuzzy implicative filter* of L if it satisfies:

(F3) $F(x \to z) \geq \min\{F(x \to (z' \to y)), F(y \to z)\}$, for all $x, y, z \in L$.

(ii) A fuzzy filter F of L is called a *fuzzy positive implicative filter* of L if it satisfies:

(F4) $F(x \to z) \geq \min\{F(x \to (y \to z)), F(x \to y)\}$, for all $x, y, z \in L$.

(iii) A fuzzy filter F of L is called a *fuzzy fantastic filter* of L if it satisfies:

(F5) $F(((x \to y) \to y) \to x) \geq \min\{F(z \to (y \to x)), F(z)\}$, for all $x, y, z \in L$.

For a fuzzy set F of L and $t \in (0, 1]$, the crisp set $U(F; t) = \{x \in L \mid F(x) \geq t\}$ is called the *level subset* of F.

Theorem 1.3 (*[11,12]*). *A fuzzy set F of L is a fuzzy (resp., implicative, positive implicative) filter of L if and only if $U(F;t)(\neq \emptyset)$ is a (resp., implicative, positive implicative) filter of L for all $t \in (0,1]$.*

By the above Theorem, we can get the following:

Theorem 1.4. *A fuzzy set F of L is a fuzzy fantastic filter of L if and only if $U(F;t)(\neq \emptyset)$ is a fantastic filter of L for all $t \in (0,1]$.*

A fuzzy set F of a BL-algebra L having the form

$$F(y) = \begin{cases} t(\neq 0) & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

is said to be *fuzzy point with support x and value t and is denoted by $U(x;t)$*. A fuzzy point $U(x;t)$ is said to *belong to* (resp. be *quasi-coincident with*) a fuzzy set F, written as $U(x;t) \in F$ (resp. $U(x;t)qF$) if $F(x) \geq t$ (resp. $F(x) + t > 1$). If $U(x;t) \in F$ or (resp. and) $U(x;t)qF$, then we write $U(x;t) \in \vee q$ (resp. $\in \wedge q$) F. The symbol $\overline{\in \vee q}$ means that $\in \vee q$ does not hold. Using the notion of "membership ($\in$)" and "quasi-coincidence (q)" of fuzzy points with fuzzy subsets, we obtain the concept of (α, β)-fuzzy subsemigroup, where α and β are any two of $\{\in, q, \in \vee q, \in \wedge q\}$ with $\alpha \neq \in \wedge q$, was introduced in [35]. It is noteworthy that the most viable generalization of Rosenfeld's fuzzy subgroup is the notion of $(\in, \in \vee q)$-fuzzy subgroup.

Definition 1.5 (*[13]*). A fuzzy set F of L is said to be an $(\in, \in \vee$ q$)$-*fuzzy filter* of L if for all $t, r \in (0,1]$ and $x, y \in L$,
(F6) $U(x;t) \in F$ and $U(y;r) \in F$ imply $U(x \odot y; \min\{t,r\}) \in \vee qF$,
(F7) $U(x;r) \in F$ implies $U(y;r) \in \vee qF$ with $x \leq y$.

Definition 1.6 (*[13]*). (i) An $(\in, \in \vee q)$-fuzzy filter F of L is called an $(\in, \in \vee q)$-*fuzzy implicative filter* of L if it satisfies:
(F8) $F(x \to z) \geq \min\{F(x \to (z' \to y)), F(y \to z), 0.5\}$, for all $x, y, z \in L$.
(ii) An $(\in, \in \vee q)$-fuzzy filter F of L is called an $(\in, \in \vee q)$-*fuzzy positive implicative filter* of L if it satisfies:
(F9) $F(x \to z) \geq \min\{F(x \to (y \to z)), F(x \to y), 0.5\}$, for all $x, y, z \in L$.
(iii) An $(\in, \in \vee q)$-fuzzy filter F of L is called an $(\in, \in \vee q)$-*fuzzy fantastic filter* of L if it satisfies:
(F10) $F(((x \to y) \to y) \to x) \geq \min\{F(z \to (y \to x)), F(z), 0.5\}$, for all $x, y, z \in L$.

Theorem 1.7 (*[13]*). *A fuzzy set F of L is an $(\in, \in \vee$ q$)$-fuzzy (resp., implicative, positive implicative, fantastic) filter of L if and only if $U(F;t)(\neq \emptyset)$ is a (resp., implicative, positive implicative, fantastic) filter of L for all $t \in (0, 0.5]$.*

2. Generalized fuzzy filters

Consider $J = \{t \mid t \in (0,1]$ and $U(F;t)$ is an empty set or a filter of $L\}$. We now consider the following questions:

(i) If $J = (0.5, 1]$, what kind of fuzzy filters of L will be F?
(ii) If $J = (\alpha, \beta]$, $(\alpha, \beta \in (0,1])$, whether F will be a kind of fuzzy filters of L or not?
(iii) Can we give a description for the relationship between the above generalized fuzzy filters?

Definition 2.1. A fuzzy set F of L is called an $(\overline{\in}, \overline{\in} \vee \overline{q})$-*fuzzy filter* of L if for all $t, r \in (0,1]$ and for all $x, y \in L$,
(F11) $U(x \odot y; \min\{t,r\}) \overline{\in} F$ implies $U(x;t) \overline{\in} \vee \overline{q} F$ or $U(y;r) \overline{\in} \vee \overline{q} F$,
(F12) $U(y;r) \overline{\in} F$ implies $U(x;r) \overline{\in} \vee \overline{q} F$ with $x \leq y$.

Example 2.2. Let $L = \{0, a, b, 1\}$, where $0 < a < b < 1$. Then we define $x \wedge y = \min\{x,y\}$, $x \vee y = \max\{x,y\}$, and $\odot$ and $\to$ as follows:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

It is clear that $(L, \wedge, \vee, \odot, \to, 1)$ is now a BL-algebra. Define a fuzzy set F in L by $F(0) = 0.2$, $F(a) = 0.5$ and $F(1) = F(b) = 0.6$. It is routine to verify that F is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy filter of L, but it could neither be a fuzzy filter of L, nor an $(\in, \in \vee q)$-fuzzy filter of L.

Theorem 2.3. *A fuzzy set F of L is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy filter of L if and only if for any $x, y \in L$,*
(F13) $\max\{F(x \odot y), 0.5\} \geq \min\{F(x), F(y)\}$,
(F14) $\max\{F(y), 0.5\} \geq F(x)$ *with $x \leq y$.*

Proof. (F11)$\Rightarrow$ (F13) If there exist $x, y \in L$ such that $\max\{F(x \odot y), 0.5\} < t = \min\{F(x), F(y)\}$, then $0.5 < t \leq 1$, $U(x \odot y; t) \overline{\in} F$ and $U(x;t) \in F$, $U(y;t) \in F$. By (F11), we have $U(x;t)\overline{q}F$ or $U(y;t)\overline{q}F$. Then, ($t \leq F(x)$ and $t + F(x) \leq 1$) or ($t \leq F(y)$ and $t + F(y) \leq 1$). Thus, $t \leq 0.5$, contradiction.

(F13)⇒ (F11) Let $U(x \odot y; \min\{t, r\}) \overline{\in} F$, then $F(x \odot y) < \min\{t, r\}$.

(a) If $F(x \odot y) \geq \min\{F(x), F(y)\}$, then $\min\{F(x), F(y)\} < \min\{t, r\}$, and consequently, $F(x) < t$ or $F(y) < r$. It follows that $U(x; t)\overline{\in}F$ or $U(y; r)\overline{\in}F$. Thus, $U(x; t)\overline{\in} \vee \overline{q}F$ or $U(y; r)\overline{\in} \vee \overline{q}F$.

(b) If $F(x \odot y) < \min\{F(x), F(y)\}$, then by (F13), we have $0.5 \geq \min\{F(x), F(y)\}$. Putting $U(x; t) \in F$ or, $U(y; r) \in F$, then $t \leq F(x) \leq 0.5$ or $r \leq F(y) \leq 0.5$. It follows that $U(x; t)\overline{q}F$ or $U(y; r)\overline{q}F$, and thus, $U(x; t)\overline{\in} \vee \overline{q}F$ or $U(y; r)\overline{\in} \vee \overline{q}F$.

(F12) ⇒ (F14) Let $x \leq y$, if there exist $x, y \in L$ such that $\max\{F(y), 0.5\} < t = F(x)$, then $0.5 < t \leq 1$, $U(y; t)\overline{\in}F$ and $U(x; t) \in F$. Since $U(y; t)\overline{\in}F$, by (F12), we have $U(x; t)\overline{q}F$. Then $t \leq F(x)$ and $t + F(x) \leq 1$, which implies, $t \leq 0.5$, contradiction.

(F14) ⇒ (F12) Let $U(y; t)\overline{\in}F$ with $x \leq y$, then $F(y) < t$.

(a) If $F(y) \geq F(x)$, then $F(x) < t$, and consequently, $U(x; t)\overline{\in}F$. Thus, $U(x; r)\overline{\in} \vee \overline{q}F$.

(b) If $F(y) < F(x)$, then by (F14), we have, $0.5 \geq F(x)$. Let $U(x; t) \in F$, then $t \leq F(x) \leq 0.5$. It follows that $U(x; t)\overline{q}F$, and thus, $U(x; r)\overline{\in} \vee \overline{q}F$.

This completes the proof. □

Lemma 2.4 (*[13]*). *Let F be a fuzzy set of L. Then* $U(F; t)(\neq \emptyset)$ *is a filter of L for all* $0.5 < t \leq 1$ *if and only if it satisfies (F13) and (F14).*

Theorem 2.5. *A fuzzy set F of L is an* $(\overline{\in}, \overline{\in} \vee \overline{q})$*-fuzzy filter of L if and only if* $U(F; t)(\neq \emptyset)$ *is a filter for all* $0.5 < t \leq 1$.

Proof. This Theorem is an immediate consequence of Theorem 2.3 and Lemma 2.4. □

Remark 2.6. Let F be a fuzzy set of a BL-algebra L and $J = \{t | t \in (0, 1]$ and $U(F; t)$ an empty subset or a filter of $L\}$.

(i) If $J = (0, 1]$, then F is an ordinary fuzzy filter of L (Theorem 1.3);

(ii) If $J = (0, 0.5]$, then F is an $(\in, \in \vee\, q)$-fuzzy filter of L (Theorem 1.7);

(iii) If $J = (0.5, 1]$, then F is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy filter of L (Theorem 2.5).

We now extend the above theory.

Definition 2.7. Given $\alpha, \beta \in (0, 1]$ and $\alpha < \beta$, we call a fuzzy set F of L a *fuzzy filter with thresholds* $(\alpha, \beta]$ of L if for all $x, y \in L$, the following conditions are satisfied:

(F15) $\max\{F(x \odot y), \alpha\} \geq \min\{F(x), F(y), \beta\}$,

(F16) $\max\{F(y), \alpha\} \geq \min\{F(x), \beta\}$ with $x \leq y$.

Theorem 2.8. *A fuzzy set F of L is a fuzzy filter with thresholds* $(\alpha, \beta]$ *of L if and only if* $U(F; t)(\neq \emptyset)$ *is a filter of L for all* $\alpha < t \leq \beta$.

Proof. The proof is similar to the proof of Lemma 2.4. □

Remark 2.9. (1) By Theorem 2.5, we have the following result: if F is a fuzzy filter with thresholds $(\alpha, \beta]$ of L, then we can conclude that

(i) F is an ordinary fuzzy implicative filter when $\alpha = 0, \beta = 1$;

(ii) F is an $(\in, \in \vee\, q)$-fuzzy filter when $\alpha = 0, \beta = 0.5$;

(iii) F is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy filter when $\alpha = 0.5, \beta = 1$.

(2) By Theorem 2.5, we can define other fuzzy filters of L, same as the fuzzy filter with thresholds (0.3, 0.9], with thresholds (0.4, 0.6] of L, etc.

(3) However, the fuzzy filter with thresholds of L may not be the usual fuzzy filter, or may not be an $(\in, \in \vee\, q)$-fuzzy filter, or may not be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy filter, respectively. These situations can be shown in the following example:

Example 2.10. Consider the BL-algebra L as in Example 2.2. Define a fuzzy set F of L by $F(0) = 0.2$, $F(a) = 0.4$, $F(b) = 0.8$ and $F(1) = 0.6$.

Then, we have

$$U(F; t) = \begin{cases} \{0, a, b, 1\} & \text{if } 0 < t \leq 0.2, \\ \{1, b, a\} & \text{if } 0.2 < t \leq 0.4, \\ \{1, b\} & \text{if } 0.4 < t \leq 0.6, \\ \{b\} & \text{if } 0.6 < t \leq 0.8, \\ \emptyset & \text{if } 0.8 < t \leq 1. \end{cases}$$

Thus, F is a fuzzy filter with thresholds (0.4, 0.6] of L. But F could neither be a fuzzy filter, an $(\in, \in \vee\, q)$-fuzzy filter of L, nor an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy filter of L.

3. Generalized fuzzy implicative (positive implicative, fantastic) filters

In this section, we divide into three parts. In Section 3.1, we describe the relationships among ordinary fuzzy implicative filters, $(\in, \in \vee\, q)$-fuzzy implicative filters and $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy implicative filters of BL-algebras. In Section 3.2, we describe the relationships among ordinary fuzzy positive implicative filters, $(\in, \in \vee\, q)$-fuzzy positive implicative filters and $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy positive implicative filters of BL-algebras. Further, the relationships among ordinary fuzzy fantastic filters, $(\in, \in \vee\, q)$-fuzzy fantastic filters and $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy fantastic filters of BL-algebras are considered in Section 3.3.

3.1. Generalized fuzzy implicative filters

Consider $J = \{t|t \in (0, 1]$ and $U(F;t)$ is an empty set or an implicative filter of $L\}$. We now consider the following questions:

(i) If $J = (0.5, 1]$, what kind of fuzzy implicative filters of L will be F?
(ii) If $J = (\alpha, \beta]$, $(\alpha, \beta \in (0, 1])$, whether F will be a kind of fuzzy implicative filters of L or not?
(iii) Can we give a description for the relationship between the above generalized fuzzy implicative filters ?

Definition 3.1.1. An $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy filter of L is called an $(\overline{\in}, \overline{\in} \vee \overline{q})$-*fuzzy implicative filter* of L if it satisfies:
(F17) $\max\{F(x \to z), 0.5\} \ge \min\{F(x \to (z' \to y)), F(y \to z)\}$, for all $x, y, z \in L$.

Example 3.1.2. Let $L = \{0, a, b, 1\}$, where $0 < a < b < 1$. Then we define $x \wedge y = \min\{x, y\}$, $x \vee y = \max\{x, y\}$, and $\odot$ and $\to$ as follows:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	a	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	b	1	1
1	0	a	b	1

.

It is clear that $(L, \wedge, \vee, \odot, \to, 1)$ is now a BL-algebra. Define a fuzzy set F of L by $F(0) = 0.2$, $F(a) = 0.8$, $F(b) = 0$ and $F(1) = 0.6$. It is routine to verify that F is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy implicative filter of L, but it could neither be a fuzzy implicative filter of L, nor an $(\in, \in \vee q)$-fuzzy implicative filter of L.

Lemma 3.1.3. *Let F be a fuzzy set of L. Then $U(F;t)(\neq \emptyset)$ is an implicative filter of L for all $0.5 < t \le 1$ if and only if it satisfies (F13), (F14) and (F17).*

Proof. Assume that $U(F;t)(\neq \emptyset)$ is an implicative filter of L. Then,it follows from Lemma 2.4 that (F13) and (F14) hold. If there exist $x, y, z \in L$ such that $\max\{F(x \to z), 0.5\} < t = \min\{F(x \to (z' \to y)), F(y \to z), 0.5\}$, then $0.5 < t \le 1$, $F(x \to z) < t$ and $x \to (z' \to y), y \to z \in U(F;t)$. Since $U(F;t)$ is an implicative filter of L, $x \to z \in U(F;t)$, and so $F(x \to z) \ge t$, which is a contradiction. Hence (F17) holds.

Conversely, suppose that the conditions (F13), (F14) and (F17) hold. Then, it follows from Lemma 2.4 that $U(F;t)$ is a filter of L. Assume that $0.5 < t \le 1$, $x \to (z' \to y), y \to z \in U(F;t)$. Then $0.5 < t \le \min\{F(x \to (z' \to y)), F(y \to z)\} \le \max\{F(x \to z), 0.5\} < F(x \to z)$, which implies that $x \to z \in U(F;t)$. Thus $U(F;t)$ is an implicative filter of L. □

Theorem 3.1.4. *A fuzzy set F of L is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy implicative filter of L if and only if $U(F;t)(\neq \emptyset)$ is an implicative filter for all $0.5 < t \le 1$.*

Proof. This Theorem is an immediate consequence of Theorem 2.5 and Lemma 3.1.3. □

Remark 3.1.5. Let F be a fuzzy set of a BL-algebra L and $J = \{t|t \in (0, 1]$ and $U(F;t)$ an empty subset or an implicative filter of $L\}$.
(i) If $J = (0, 1]$, then F is an ordinary fuzzy implicative filter of L (Theorem 1.3);
(ii) If $J = (0, 0.5]$, then F is an $(\in, \in \vee \mathrm{q})$-fuzzy implicative filter of L (Theorem 1.7);
(iii) If $J = (0.5, 1]$, then F is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy implicative filter of L (Theorem 3.1.4).

We now extend the above theory.

Definition 3.1.6. Given $\alpha, \beta \in (0, 1]$ and $\alpha < \beta$, we call a fuzzy set F of L a *fuzzy implicative filter with thresholds* $(\alpha, \beta]$ of L if it satisfies (F15),(F16) and
(F18) $\max\{F(x \to z), \alpha\} \ge \min\{F(x \to (z' \to y)), F(y \to z), \beta\}$, for all $x, y, z \in L$.

Theorem 3.1.7. *A fuzzy set F of L is a fuzzy implicative filter with thresholds $(\alpha, \beta]$ of L if and only if $U(F;t)(\neq \emptyset)$ is an implicative filter of L for all $\alpha < t \le \beta$.*

Proof. The proof is similar to the proof of Theorem 3.1.4. □

Remark 3.1.8. (1) By Definition 3.1.6, we have the following result: if F is a fuzzy implicative filter with thresholds $(\alpha, \beta]$ of L, then we can conclude that
(i) F is an ordinary fuzzy implicative filter when $\alpha = 0, \beta = 1$;
(ii) F is an $(\in, \in \vee \mathrm{q})$-fuzzy implicative filter when $\alpha = 0, \beta = 0.5$;
(iii) F is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy implicative filter when $\alpha = 0.5, \beta = 1$.
(2) By Definition 3.1.6, we can define other fuzzy implicative filters of L, similarly to the fuzzy implicative filter with thresholds (0.3, 0.9], with thresholds (0.4, 0.6] of L, etc.
(3) However, the fuzzy implicative filter with thresholds of L may not be the usual fuzzy implicative filter, or may not be an $(\in, \in \vee \mathrm{q})$-fuzzy implicative filter, or may not be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy implicative filter, respectively. These situations can

be shown in the following example:

Example 3.1.9. Consider the BL-algebra L as in Example 3.1.2. Define a fuzzy set F of L by $F(0) = 0.4$, $F(a) = 0.8$, $F(b) = 0.2$ and $F(1) = 0.6$.

Then, we have

$$U(F;t) = \begin{cases} \{0, a, b, 1\} & \text{if } 0 < t \leq 0.2, \\ \{1, 0, a\} & \text{if } 0.2 < t \leq 0.4, \\ \{1, a\} & \text{if } 0.4 < t \leq 0.6, \\ \{a\} & \text{if } 0.6 < t \leq 0.8, \\ \emptyset & \text{if } 0.8 < t \leq 1. \end{cases}$$

Thus, F is a fuzzy implicative filter with thresholds $(0.4, 0.6]$ of L. But F could neither be a fuzzy implicative filter, an $(\in, \in \vee \mathrm{q})$-fuzzy implicative filter of L, nor an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy implicative filter of L.

3.2. Generalized fuzzy positive implicative filters

Consider $J = \{t | t \in (0, 1]$ and $U(F; t)$ is an empty set or a positive implicative filter of $L\}$. We now consider the following questions:

(i) If $J = (0.5, 1]$, what kind of fuzzy positive implicative filters of L will be F?
(ii) If $J = (\alpha, \beta]$, $(\alpha, \beta \in (0, 1])$, whether F will be a kind of fuzzy positive implicative filters of L or not?
(iii) Can we give a description for the relationship between the above generalized fuzzy positive implicative filters?

Definition 3.2.1. An $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy filter of L is called an $(\overline{\in}, \overline{\in} \vee \overline{q})$-*fuzzy positive implicative filter* of L if it satisfies:
(F19) $\max\{F(x \to z), 0.5\} \geq \min\{F(x \to (y \to z)), F(x \to y)\}$, for all $x, y, z \in L$.

Example 3.2.2. Let $L = \{0, a, b, c, 1\}$, where $0 < a < b < c < 1$. Then we define $x \wedge y = \min\{x, y\}$, $x \vee y = \max\{x, y\}$, and $\odot$ and $\to$ as follows:

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	a	b
c	0	a	a	c	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	c	1	c	1
c	0	b	b	1	1
1	0	a	b	c	1

It is clear that $(L, \wedge, \vee, \odot, \to, 1)$ is now a BL-algebra. Define a fuzzy set F of L by $F(0) = F(c) = 0.2$, $F(a) = 0.4$, $F(b) = 0.6$ and $F(1) = 0.8$. It is routine to verify that F is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy positive implicative filter of L, but it could neither be a fuzzy positive implicative filter of L, nor an $(\in, \in \vee q)$-fuzzy positive implicative filter of L.

Lemma 3.2.3. *Let F be a fuzzy set of L. Then $U(F;t)(\neq \emptyset)$ is a positive implicative filter of L for all $0.5 < t \leq 1$ if and only if it satisfies (F13), (F14) and (F19).*

Proof. It is similar to Lemma 3.1.3. □

Theorem 3.2.4. *A fuzzy set F of L is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy positive implicative filter of L if and only if $U(F;t)(\neq \emptyset)$ is a positive implicative filter for all $0.5 < t \leq 1$.*

Proof. This theorem is an immediate consequence of Theorem 2.5 and Lemma 3.2.3. □

Remark 3.2.5. Let F be a fuzzy set of a BL-algebra L and $J = \{t | t \in (0, 1]$ and $U(F; t)$ an empty subset or a positive implicative filter of $L\}$.
(i) If $J = (0, 1]$, then F is an ordinary fuzzy positive implicative filter of L (Theorem 1.3);
(ii) If $J = (0, 0.5]$, then F is an $(\in, \in \vee \mathrm{q})$-fuzzy positive implicative filter of L (Theorem 1.7);
(iii) If $J = (0.5, 1]$, then F is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy positive implicative filter of L (Theorem 3.2.4).

We now extend the above theory.

Definition 3.2.6. Given $\alpha, \beta \in (0, 1]$ and $\alpha < \beta$, we call a fuzzy set F of L a *fuzzy positive implicative filter with thresholds* $(\alpha, \beta]$ of L if it satisfies (F15),(F16) and
(F20) $\max\{F(x \to z), \alpha\} \geq \min\{F(x \to (y \to z)), F(x \to y), \beta\}$, for all $x, y, z \in L$.

Theorem 3.2.7. *A fuzzy set F of L is a fuzzy positive implicative filter with thresholds $(\alpha, \beta]$ of L if and only if $U(F;t)(\neq \emptyset)$ is a positive implicative filter of L for all $\alpha < t \leq \beta$.*

Proof. The proof is similar to the proof of Theorem 3.2.4. □

Remark 3.2.8. (1) By Definition 3.2.6, we have the following result: if F is a fuzzy positive implicative filter with thresholds $(\alpha, \beta]$ of L, then we can conclude that

(i) F is an ordinary fuzzy positive implicative filter when $\alpha = 0$, $\beta = 1$;
(ii) F is an $(\in, \in \vee \mathrm{q})$-fuzzy positive implicative filter when $\alpha = 0$, $\beta = 0.5$;
(iii) F is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy positive implicative filter when $\alpha = 0.5$, $\beta = 1$.

(2) By Definition 3.2.6, we can define other fuzzy positive implicative filters of L, similarly to the fuzzy positive implicative filter with thresholds $(0.3, 0.9]$, with thresholds $(0.4, 0.6]$ of L, etc.

(3) However, the fuzzy positive implicative filter with thresholds of L may not be the usual fuzzy positive implicative filter, or may not be an $(\in, \in \vee \mathrm{q})$-fuzzy positive implicative filter, or may not be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy positive implicative filter, respectively. These situations can be shown in the following example:

Example 3.2.9. Consider the BL-algebra L as in Example 3.2.2. Define a fuzzy set F of L by $F(0) = F(c) = 0.2$, $F(a) = 0.4$, $F(b) = 0.8$ and $F(1) = 0.6$.

Then, we have

$$U(F;t) = \begin{cases} \{0, a, b, c, 1\} & \text{if } 0 < t \leq 0.2, \\ \{1, a, b\} & \text{if } 0.2 < t \leq 0.4, \\ \{1, b\} & \text{if } 0.4 < t \leq 0.6, \\ \{b\} & \text{if } 0.6 < t \leq 0.8, \\ \emptyset & \text{if } 0.8 < t \leq 1. \end{cases}$$

Thus, F is a fuzzy positive implicative filter with thresholds $(0.4, 0.6]$ of L. But F could neither be a fuzzy positive implicative filter, an $(\in, \in \vee \mathrm{q})$-fuzzy positive implicative filter of L, nor an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy positive implicative filter of L.

3.3. Generalized fuzzy fantastic filters

Consider $J = \{t | t \in (0, 1]$ and $U(F; t)$ is an empty set or a fantastic filter of $L\}$. We now consider the following questions:

(i) If $J = (0.5, 1]$, what kind of fuzzy fantastic filters of L will be F?
(ii) If $J = (\alpha, \beta]$, $(\alpha, \beta \in (0, 1])$, whether F will be a kind of fuzzy fantastic filters of L or not?
(iii) Can we give a description for the relationship between the above generalized fuzzy fantastic filters?

Definition 3.3.1. An $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy filter of L is called an *$(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy fantastic filter* of L if it satisfies:

(F21) $\max\{F(((x \to y) \to y) \to x), 0.5\} \geq \min\{F(z \to (y \to x)), F(z)\}$, for all $x, y, z \in L$.

Example 3.3.2. Let $L = \{0, a, b, 1\}$, where $0 < a < b < 1$. Then we define $x \wedge y = \min\{x, y\}$, $x \vee y = \max\{x, y\}$, and $\odot$ and $\to$ as follows:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	a	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1

It is clear that $(L, \wedge, \vee, \odot, \to, 1)$ is now a BL-algebra. Define a fuzzy set F of L by $F(a) = 0.5$, $F(b) = F(0) = 0.2$ and $F(1) = 0.8$. It is routine to verify that F is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy fantastic filter of L, but it could neither be a fuzzy fantastic filter of L, nor an $(\in, \in \vee q)$-fuzzy fantastic filter of L.

Lemma 3.3.3. *Let F be a fuzzy set of L. Then $U(F; t)(\neq \emptyset)$ is a fantastic filter of L for all $0.5 < t \leq 1$ if and only if it satisfies (F13),(F14) and (F21).*

Proof. It is similar to Lemma 3.2.3. □

Theorem 3.3.4. *A fuzzy set F of L is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy fantastic filter of L if and only if $U(F; t)(\neq \emptyset)$ is a fantastic filter for all $0.5 < t \leq 1$.*

Proof. This Theorem is an immediate consequence of Theorem 2.5 and Lemma 3.3.3. □

Remark 3.3.5. Let F be a fuzzy set of a BL-algebra L and $J = \{t | t \in (0, 1]$ and $U(F; t)$ an empty subset or a fantastic filter of $L\}$.

(i) If $J = (0, 1]$, then F is an ordinary fuzzy fantastic filter of L (Theorem 1.4);
(ii) If $J = (0, 0.5]$, then F is an $(\in, \in \vee \mathrm{q})$-fuzzy positive implicative filter of L (Theorem 1.7);
(iii) If $J = (0.5, 1]$, then F is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy fantastic filter of L (Theorem 3.3.4).

We now extend the above theory.

Definition 3.3.6. Given $\alpha, \beta \in (0, 1]$ and $\alpha < \beta$, we call a fuzzy set F of L a *fuzzy fantastic filter with thresholds $(\alpha, \beta]$* of L if it satisfies (F15),(F16) and

(F22) $\max\{F(((x \to y) \to y) \to x), \alpha\} \geq \min\{F(z \to (y \to x)), F(z), \beta\}$, for all $x, y, z \in L$.

Theorem 3.3.7. *A fuzzy set F of L is a fuzzy fantastic filter with thresholds $(\alpha, \beta]$ of L if and only if $U(F; t)(\neq \emptyset)$ is a fantastic filter of L for all $\alpha < t \leq \beta$.*

Proof. The proof is similar to the proof of Theorem 3.3.4. □

Remark 3.3.8. (1) By Definition 3.3.6, we have the following result: if F is a fuzzy fantastic filter with thresholds $(\alpha, \beta]$ of L, then we can conclude that

(i) F is an ordinary fuzzy fantastic filter when $\alpha = 0, \beta = 1$;

(ii) F is an $(\in, \in \vee\, q)$-fuzzy fantastic filter when $\alpha = 0, \beta = 0.5$;

(iii) F is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy fantastic filter when $\alpha = 0.5, \beta = 1$.

(2) By Definition 3.3.6, we can define other fuzzy fantastic filters of L, similarly to the fuzzy fantastic filter with thresholds (0.3, 0.9], with thresholds (0.4, 0.6] of L, etc.

(3) However, the fuzzy fantastic filter with thresholds of L may not be the usual fuzzy fantastic filter, or may not be an $(\in, \in \vee\, q)$-fuzzy fantastic filter, or may not be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy fantastic filter, respectively. These situations can be shown in the following example:

Example 3.3.9. Consider the *BL*-algebra L as in Example 3.1.2. Define a fuzzy set F of L by $F(a) = 0.8$, $F(0) = 0$, $F(b) = 0.2$ and $F(1) = 0.6$.

Then, we have

$$U(F; t) = \begin{cases} \{a, b, 1\} & \text{if } 0 < t \leq 0.2, \\ \{1, a\} & \text{if } 0.2 < t \leq 0.6, \\ \{a\} & \text{if } 0.6 < t \leq 0.8, \\ \emptyset & \text{if } 0.8 < t \leq 1. \end{cases}$$

Thus, F is a fuzzy fantastic filter with thresholds (0.2, 0.6] of L. But F could neither be a fuzzy fantastic filter, an $(\in, \in \vee\, q)$-fuzzy fantastic filter of L, nor an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy fantastic filter of L.

4. Relationships among these generalized fuzzy filters

In this section, we discuss the relationships among these generalized fuzzy filters of *BL*-algebras and obtain an important result.

Lemma 4.1 ([12]). *Every implicative filter of L is a positive implicative filter.*

Lemma 4.2 ([12]). *Let A be a filter of L. Then A is an implicative filter of L if and only if $(x \to y) \to x \in A \Rightarrow x \in A$, for all $x, y \in L$.*

By the definition of fantastic filters of L, we can immediately get the following:

Lemma 4.3. *Let A be a filter of L. Then A is a fantastic filter if and only if $y \to x \in A \Rightarrow ((x \to y) \to y) \to x \in A$, for all $x, y \in L$.*

Lemma 4.4 ([12]). *Let A be a filter of L. Then A is a positive implicative filter of L if and only if $x \to (x \to y) \in A \Rightarrow x \to y \in A$, for all $x, y \in L$.*

Lemma 4.5. *Every implicative filter of L is a fantastic filter.*

Proof. Let A be an implicative filter of L. For any $x, y \in L$ be such that $y \to x \in A$. Since $x \odot ((x \to y) \to y) \leq x$, and so $x \leq ((x \to y) \to y) \leq x$, which implies, $(((x \to y) \to y) \to x \leq x \to y)$.

Thus, $((((x \to y) \to y) \to x) \to y) \to (((x \to y) \to y) \to x)$
$\geq (x \to y) \to (((x \to y) \to y) \to x)$
$\geq ((x \to y) \to y) \to ((x \to y) \to x)$
$\geq y \to x$.

By hypothesis, we have $((((x \to y) \to y) \to x) \to y) \to (((x \to y) \to y) \to x) \in A$. It follows from Lemma 4.2 that $(x \to y) \to x \in A$. This proves that $y \to x \in A \Rightarrow ((x \to y) \to y) \to x \in A$. Thus, by Lemma 4.3, we know A is a fantastic filter of L. □

Theorem 4.6. *A non-empty subset A of L is an implicative filter of L if and only if it is both a positive implicative filter and a fantastic filter.*

Proof. Necessity: Lemmas 4.1 and 4.5.

Sufficiency: Let $x, y \in L$ be such that $(x \to y) \to x \in A$. Since $(x \to y) \to x \leq (x \to y) \to ((x \to y) \to y)$, we have $((x \to y) \to y) \in A$. Since A is a positive implicative filter of L, by Lemma 4.4, we have

$$(x \to y) \to y \in A. \tag{*}$$

Since $(x \to y) \to x \leq y \to x$, we have $y \to x \in A$. By Lemma 4.3, we have

$$((x \to y) \to y) \to x \in A. \tag{**}$$

By (*) and (**), we have $x \in A$ since A is a filter of L.

This proves that $(x \to y) \to x \in A \Rightarrow x \in A$. It follows from Lemma 4.2 that A is an implicative filter of L. □

Corollary 4.7. *A non-empty subset $U(F; t)$ of L is an implicative filter of L if and only if it is both a positive implicative filter and a fantastic filter for all $t \in (0.5, 1]$.*

Finally, we give the relationships among $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy implicative filters, $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy positive implicative filters and $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy fantastic filters of *BL*-algebras.

Theorem 4.8. *A fuzzy set F of L is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy implicative filter of L if and only if it is both an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy positive implicative filter and an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy fantastic filter.*

Proof. Let F be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy implicative filter of L. By Theorem 3.1.4, we know non-empty subset $U(F; t)$ is an implicative filter of L for all $t \in (0.5, 1]$. By Corollary 4.7, $U(F; t)$ is both a positive implicative filter and a fantastic filter of L for all $t \in (0.5, 1]$. It follows from Theorems 3.2.4 and 3.3.4 that F is both an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy positive implicative filter and an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy fantastic filter of L.

Conversely, assume that F is both an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy positive implicative filter and an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy fantastic filter of L. By Theorems 3.2.4 and 3.3.4, we know non-empty subset $U(F; t)$ is both a positive implicative filter and a fantastic filter of L for all $t \in (0.5, 1]$. By Corollary 4.7, $U(F; t)$ is an implicative filter of L for all $t \in (0.5, 1]$. It follows from Theorem 3.1.4 that F is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy implicative filter of L. □

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